



TITLE:

# The Liftings of Product Form on Measure Spaces (無限次元空間のテンソル積)

AUTHOR(S):

MURAKAMI, KIYOSHI

---

CITATION:

MURAKAMI, KIYOSHI. The Liftings of Product Form on Measure Spaces (無限次元空間のテンソル積). 数理解析研究所講究録 1975, 228: 1-15

ISSUE DATE:

1975-03

URL:

<http://hdl.handle.net/2433/105410>

RIGHT:

# THE LIFTINGS OF PRODUCT FORM ON MEASURE SPACES

by Kiyoshi Murakami

## ABSTRACT

In this paper we shall study about the following concept.  
Let  $(X, \mathcal{B}_X, \mu_X)$  and  $(Y, \mathcal{B}_Y, \mu_Y)$  be two measure spaces with liftings  $\rho_X$  and  $\rho_Y$  respectively. We denote by  $(Z, \mathcal{B}_Z, \mu_Z)$  the product measure space of  $X$  and  $Y$ . Now we call a lifting  $\rho_Z$  on  $Z$  of product form  $\rho_X \times \rho_Y$  if it satisfies the following conditions

$$\rho_Z(A \times B) = \rho_X(A) \times \rho_Y(B) \quad \text{for every } A \in \mathcal{B}_X \text{ and } B \in \mathcal{B}_Y.$$

We shall show some properties the liftings of product form and some results connected with the strong lifting property.

## §0. Introduction

The theory of lifting on measure spaces has been developed for the last fifteen years. In this paper we consider the liftings on product measure spaces. In §1 we write notations and preliminary results which are used in the following sections. In §2 we give the definitions and the characterizations of the lifting of product form. In §3 we show some necessary and sufficient conditions of the existence of the lifting of product form.

The author wishes to express his hearty thanks to Professor H. Umegaki for many kind suggestions and advices in the course of preparing the present paper.

## §1. Preliminaries

Recently A. and C. Ionescu Tulcea have shown many results about the lifting on measure spaces. We shall write several fundamental notations and theorems in their book [I.T] (Topics in the theory of lifting), which play many important role in this paper.

Let  $(X, \mathcal{B}, \mu)$  be measure space and we use the following notations.  $\mathcal{B}_0 = \{A \in \mathcal{B} : \mu(A) < \infty\}$ ,  $\mathcal{B}^+ = \{A \in \mathcal{B} : \mu(A) > 0\}$ ,  $\mathcal{B}_0^+ = \mathcal{B}_0 \cap \mathcal{B}^+$  and  $\mathcal{N}_{\mathcal{B}} = \{A \in \mathcal{B} : \mu(A) = 0\}$ . We denote by  $M_R^{\infty}(X)$  the algebra of all bounded real-valued  $\mathcal{B}$ -measurable functions on  $X$ . When for  $f, g \in M_R^{\infty}(X)$  denote by  $f \equiv g$  iff  $\{x \in X : f(x) \neq g(x)\} \in \mathcal{N}_{\mathcal{B}}$ . Also for  $A, B \in \mathcal{B}$  denote by  $A \equiv B$  iff  $A \Delta B \in \mathcal{N}_{\mathcal{B}}$ .

Let  $\rho$  be a mapping of  $M_R^{\infty}(X)$  into itself and consider the following conditions,

- (I)  $\rho(f) \equiv f$ ,
- (II)  $f \equiv g$  implies  $\rho(f) = \rho(g)$ ,
- (III)  $\rho(1) = 1$ ,
- (IV)  $f \geq 0$  implies  $\rho(f) \geq 0$ ,

$$(V) \quad \rho(\alpha f + \beta g) = \alpha \rho(f) + \beta \rho(g),$$

$$(VI) \quad \rho(fg) = \rho(f) \cdot \rho(g).$$

We call  $\rho$  is a (linear) lifting of  $M_R^\infty(X)$  if it satisfies the conditions ((I) - (V)) (I) - (VI).

Let  $\theta$  be a mapping of  $\mathcal{B}$  into itself and consider the following conditions,

$$(I') \quad \theta(A) \equiv A,$$

$$(II') \quad A \equiv B \text{ implies } \theta(A) = \theta(B),$$

$$(III') \quad \theta(X) = X \text{ and } \theta(\phi) = \phi,$$

$$(IV') \quad \theta(A \cap B) = \theta(A) \cap \theta(B),$$

$$(V') \quad \theta(A \cup B) = \theta(A) \cup \theta(B).$$

We call  $\theta$  is a lower density of  $\mathcal{B}$  if it satisfies the conditions (I') - (IV') and a lifting of  $\mathcal{B}$  if it satisfies the conditions (I') - (V'). There exists a one-to-one correspondence between the liftings of  $M_R^\infty(X)$  and the liftings of  $\mathcal{B}$  which are connected with the equations.

$$(L) \quad \rho(1_A) = 1_{\theta(A)} \quad \text{for every } A \in \mathcal{B},$$

where  $1_A$  denotes the characteristic function of the set  $A$  ([I.T], Ch.3). So we use the same symbol  $\rho$  for a lifting of  $M_R^\infty(X)$  and a lifting of  $\mathcal{B}$  which are connected with the equations (L) and say it a lifting on  $X$ . When  $X$  is a topological space with a topology  $\mathcal{T}$ , we denote by  $C_R^\infty(X, \mathcal{T})$  the algebra of all bounded real-valued  $\mathcal{T}$ -continuous functions on  $X$ . A (linear) lifting  $\rho$  of  $M_R^\infty(X)$  is called strong on  $(X, \mathcal{T})$  if  $C_R^\infty(X, \mathcal{T}) \subset M_R^\infty(X)$  and it satisfies the following conditions,

$$(VII) \quad \rho(f) = f \quad \text{for every } f \in C_R^\infty(X, \mathcal{T}).$$

We shall introduce some results given in [I.T], [2], [4] and [6] which are used throughout this paper. From now on we shall assume that every measure space  $(X, \mathcal{B}, \mu)$  is complete, semi-finite localizable measure space and  $\mathcal{B}$  is identical with the  $\sigma$ -algebra

of all  $\mu$ -measurable subsets of  $X$ . We say that  $(X, \mathcal{B}, \mu)$  has the direct sum property if there exists a disjoint family  $\{X_i\}_{i \in I}$  in  $\mathcal{B}^*$  such that  $X = \bigcup_{i \in I} X_i$  and for every  $A \in \mathcal{B}^*$  there exists  $i \in I$  such that  $A \cap X_i \in \mathcal{B}^*$ . It was proved that  $(X, \mathcal{B}, \mu)$  has the direct sum property iff there exists a lifting on  $X$  ([I.T] and [6]). A derivation basis on  $X$  is a family  $\mathcal{F} = \{\mathcal{F}(x)\}_{x \in X}$ , where for  $x \in X$   $\mathcal{F}(x)$  is a filter basis on  $\mathcal{B}^*$ . For  $A \in \mathcal{B}$  and  $x \in X$  define

$$(D) \quad D(A, \mathcal{F})(x) = \lim_{\mathcal{F}(x)} \mu(A \cap B) / \mu(B)$$

if the limit of the right side exists. A derivation basis  $\mathcal{F}$  is said to be weak if for every  $A \in \mathcal{B}$  the set  $N_{\mathcal{F}}^A$  belongs to  $\mathcal{N}_{\mathcal{B}}$ , where  $N_{\mathcal{F}}^A$  is the set of all  $x \in X$  such that  $D(A, \mathcal{F})(x)$  isn't defined or  $D(A, \mathcal{F})(x) \neq 1_A(x)$ . Let  $\mathcal{F}$  be a weak derivation basis on  $X$ . For  $A \in \mathcal{B}$ , put  $\theta_{\mathcal{F}}(A) = \{x \in X : D(A, \mathcal{F})(x) = 1\}$ , then  $\theta_{\mathcal{F}}$  is a lower density of  $\mathcal{B}$  ([4] and [5]). Let  $L_R^\infty(X) = M_R^\infty(X) / \equiv$ ,  $B(X) = \mathcal{B} / \equiv$  and denote by  $f \mapsto \dot{f}$  (respectively  $A \mapsto \dot{A}$ ) the canonical mapping of  $M_R^\infty(X)$  onto  $L_R^\infty(X)$  (respectively  $\mathcal{B}$  onto  $B(X)$ ). Since  $X$  is localizable,  $B(X)$  is a complete Boolean algebra and by  $\dot{A} \mapsto \dot{1}_A$  it is isomorphic to the Boolean algebra of all idempotents in  $L_R^\infty(X)$ . Denote by  $\check{X}$  the Stone representation space of  $B(X)$  i.e.  $\check{X}$  is the set of all maximal ideals of  $B(\check{X})$  with a totally and extremally disconnected compact hausdorff topology and  $B(X)$  is isomorphic to the Boolean algebra of all clopen subsets of  $\check{X}$ . Since  $C_R(\check{X})$  and  $L_R^\infty(X)$  are generated by the Boolean algebra of all their idempotents,  $C_R(\check{X})$  is isomorphic to  $L_R^\infty(X)$  and hence  $\check{X}$  is also the Gelfand representation space of  $L_R^\infty(X)$ . It was proved that a lifting on  $X$  induces a mapping from  $X$  into  $\check{X}$  ([I.T] and [2]). Let  $\rho$  be a lifting on  $X$ . For  $x \in X$ , put  $\mathcal{J}_\rho(x) = \{\dot{A} \in B(X) : x \in \rho(A)^c\}$ . It is easy to show that  $\mathcal{J}_\rho(x) \in \check{X}$ . We note that for  $x \in X$  and  $\dot{f} \in L_R^\infty(X)$  put  $\chi_x(\dot{f}) = \rho(f)(x)$  then  $\chi_x$  is a character of  $L_R^\infty(X)$  and the corresponding maximal ideal  $\{\dot{f} \in L_R^\infty(X) : \chi_x(\dot{f}) = 0\}$  is generated by  $\{\dot{1}_A :$

$\dot{A} \in \mathcal{J}_\rho(x)\}$ . Now we define a mapping  $\mathcal{J}_\rho(\cdot)$  from  $X$  into  $\check{X}$ . Denote by  $\mathcal{T}_\rho$  the weak topology induced by  $\mathcal{J}_\rho(\cdot)$ . Since  $\check{X}$  has a base  $\{E(\dot{A}) : A \in \mathcal{B}\}$ , where  $E(\dot{A}) = \{\dot{x} \in \check{X} : \dot{A} \in \mathcal{J}(\dot{x})\}$  and  $\mathcal{J}_\rho^{-1}(E(\dot{A}^c)) = \rho(A)$ ,  $\mathcal{T}_\rho$  has a base  $\{\rho(A) : A \in \mathcal{B}\}$ . It was shown that  $\mathcal{T}_\rho \subset \mathcal{B}$ ,  $C_R^\infty(X, \mathcal{T}_\rho) = \{\rho(f) : f \in M_R^\infty(X)\}$  and  $C_R^\infty(X, \mathcal{T}_\rho)$  is isomorphic to  $L_R^\infty(X)$  by  $f \rightarrow \dot{f}$  ([I.T], Ch.5). Next we consider a mapping from  $X$  into  $\check{X}$  and construct a lifting from it under suitable conditions.

LEMMA 1. Let  $\mathcal{J}(\cdot)$  be a mapping from  $X$  into  $\check{X}$ . For  $A \in \mathcal{B}$ , put  $\rho_A(A) = \{x \in X : \dot{A}^c \in \mathcal{J}(x)\}$ . Then  $\rho_A$  is a mapping from  $\mathcal{B}$  into  $\mathcal{P}(X)$ , where  $\mathcal{P}(X)$  is a family of all subsets of  $X$ , and  $\rho_A$  satisfies the conditions (II') - (V') of a lifting.

Proof. It is clear that  $\rho_A$  satisfies (II') - (IV'). By (IV') we have  $\rho_A(A) \cup \rho_A(B) \subset \rho_A(A \cup B)$ . If  $x \in \rho_A(A \cup B)$  then  $(A \cup B)^c = \dot{A}^c \cap \dot{B}^c \in \mathcal{J}(x)$ . Since  $\mathcal{J}(x)$  is maximal,  $\dot{A}^c \in \mathcal{J}(x)$  or  $\dot{B}^c \in \mathcal{J}(x)$  and we have then  $x \in \rho_A(A)$  or  $x \in \rho_A(B)$ . So  $\rho_A$  satisfies (V').

q.e.d.

The above lemma implies the following. If  $\rho_A$  satisfies the condition (I') then  $\rho_A$  is a lifting on  $X$ . We shall give a condition that  $\rho_A$  satisfies the condition (I').

PROPOSITION 1. Let  $\theta$  be a lower density of  $\mathcal{B}$ . For  $x \in X$ , put  $\mathcal{J}_\theta(x) = \{\dot{A} \in \mathcal{B}(X) : x \in \theta(A^c)\}$  then  $\mathcal{J}_\theta(x)$  is a proper ideal of  $\mathcal{B}(X)$ . Let  $N_0 \in \mathcal{N}_\theta$ . If  $\mathcal{J}(x) \supset \mathcal{J}_\theta(x)$  for every  $x \in X - N_0$  then  $\theta(A) - N_0 \subset \rho_A(A)$  for every  $A \in \mathcal{B}$ . In this case  $\rho_A$  satisfies the condition (I').

Proof. It is clear that  $\mathcal{J}_\theta(x)$  is a proper ideal of  $\mathcal{B}(X)$ . Now we assume  $\mathcal{J}(\cdot)$  satisfies the conditions. If  $x \in \theta(A) - N_0$  then  $\dot{A}^c \in \mathcal{J}_\theta(x) \subset \mathcal{J}(x)$  and hence  $x \in \rho_A(A)$ . So  $\theta(A) - N_0 \subset \rho_A(A)$  for every  $A \in \mathcal{B}$ . Since  $\rho_A(A) = \rho_A(A^c)^c \subset \theta(A^c)^c \cup N_0$  and  $A \equiv \theta(A) - N_0 \equiv \theta(A^c)^c \cup N_0$  we have  $\rho_A(A) \equiv A$ .

q.e.d.

By Zorn's lemma every proper ideal of  $B(X)$  is contained in a maximal ideal, the existence of a lower density of  $\mathfrak{B}$  implies the existence of a mapping from  $X$  into  $\tilde{X}$  which satisfies the conditions of above proposition, i.e. the existence of a lower density of  $\mathfrak{B}$  implies the existence of a lifting.

REMARK. We note that  $\{l_A : A \in \mathcal{Q}_\theta(X)\}$  generates a closed proper ideal of  $L_R^\infty(X)$  and it coincides with  $J_X$  in [I.T], Ch.5, Proposition 2. Moreover if  $\mathcal{J}(\cdot)$  and  $N_0 \in \mathcal{N}_\mathfrak{B}$  is same in above proposition then for  $x \in X - N_0$ ,  $\{l_A : A \in \mathcal{J}(x)\}$  generates a maximal ideal of  $L_R^\infty(X)$  and the character  $\chi_x$  of  $L_R^\infty(X)$  corresponding to it vanishes on  $J_X$ . Consequently without the topology  $\mathfrak{T}_\theta$  ([I.T], Ch.5), we can construct same lifting in [I.T], Ch.5, Proposition 2.

## §2. The lifting of product form

Let  $(X, \mathfrak{B}_X, \mu_X)$  and  $(Y, \mathfrak{B}_Y, \mu_Y)$  be measure spaces with liftings  $\rho_X$  and  $\rho_Y$  respectively. Let  $(Z, \mathfrak{B}_Z, \mu_Z)$  be a measure space such that  $Z = X \times Y$  and  $\mathfrak{B}_Z$  is the  $\sigma$ -algebra of all  $\mu_X \times \mu_Y$ -measurable subsets of  $Z$ , where  $\mu_X \times \mu_Y$  is the product measure of  $\mu_X$  and  $\mu_Y$  in the sense of [1], Ch.6 and  $\mu_Z$  is the contracted measure of  $\widehat{\mu_X \times \mu_Y}$  which is the extension of  $\mu_X \times \mu_Y$  on  $\mathfrak{B}_Z$ , i.e.

$$\mu_Z(A) = \sup \{ \widehat{\mu_X \times \mu_Y}(B) : B \in \mathfrak{B}_Z, B \subset A \text{ and } \widehat{\mu_X \times \mu_Y}(B) < \infty \}.$$

In the following we shall write  $X, Y$  and  $Z$  instead of  $(X, \mathfrak{B}_X, \mu_X)$ , and  $(Y, \mathfrak{B}_Y, \mu_Y)$  and  $(Z, \mathfrak{B}_Z, \mu_Z)$  respectively. We note if  $X$  and  $Y$  are  $\sigma$ -finite then  $Z$  is the usual completion of the product measure space  $(X \times Y, \mathfrak{B}_X \times \mathfrak{B}_Y, \mu_X \times \mu_Y)$ .

LEMMA 2.  $Z$  is strictly localizable i.e. a semi-finite and complete measure space with the direct sum property.

Proof. The semi-finiteness and completeness follows from the construction. Since there exist liftings on  $X$  and  $Y$ ,  $X$  and  $Y$  have the direct sum Property. Let  $\{X_i\}_{i \in I}$  (respectively  $\{Y_k\}_{k \in K}$ )

be a disjoint family in  $\mathcal{B}_X^*$  (respectively  $\mathcal{B}_Y^*$ ) by which the direct sum property is satisfied. It is clear that  $\{X_\lambda \times Y_\kappa\}_{(\lambda, \kappa) \in I \times K}$  is a disjoint family in  $\mathcal{B}_Z^*$  and  $Z = \bigcup_{(\lambda, \kappa) \in I \times K} X_\lambda \times Y_\kappa$ . For  $A \in \mathcal{B}_Z^*$  there exists  $A' \in \mathcal{B}_X \times \mathcal{B}_Y$  and  $N \in \mathcal{N}_Z$  such that  $A \subset A' \cup N$ ,  $A' \cap N = \emptyset$  and  $\mu_Z(A) = \mu_X \times \mu_Y(A')$ . If  $\mu_Z(A \cap (X_\lambda \times Y_\kappa)) = 0$  for every  $(\lambda, \kappa) \in I \times K$  then  $\mu_X \times \mu_Y(A' \cap (X_\lambda \times Y_\kappa)) = 0$  for every  $(\lambda, \kappa) \in I \times K$ . From the definition of  $\mu_X \times \mu_Y$  we have then  $\mu_X \times \mu_Y(A') = 0$ . This contradicts to  $\mu_X \times \mu_Y(A') = \mu_Z(A) > 0$ .

q.e.d.

This lemma implies that there always exist liftings on  $Z$ . So we consider the following definition.

DEFINITION. A lifting  $\rho_Z$  on  $Z$  is said to be of product form  $\rho_X \times \rho_Y$  if it satisfies the following conditions,

$$(P) \quad \rho_Z(A \times B) = \rho_X(A) \times \rho_Y(B) \quad \text{for every } A \in \mathcal{B}_X \text{ and } B \in \mathcal{B}_Y.$$

We shall show some characterizations of the lifting of product form  $\rho_X \times \rho_Y$ .

THEOREM. Let  $\rho_X$ ,  $\rho_Y$  and  $\rho_Z$  be liftings on  $X$ ,  $Y$  and  $Z$  respectively. Then for these  $\rho_X$ ,  $\rho_Y$  and  $\rho_Z$  the following conditions are equivalent to each others.

- i)  $\rho_Z$  is of product form  $\rho_X \times \rho_Y$ ,
- ii)  $\rho_Z(A \times Y) \subset \rho_X(A) \times Y$  and  $\rho_Z(X \times B) \subset X \times \rho_Y(B)$  for  $A \in \mathcal{B}_X$  and  $B \in \mathcal{B}_Y$ ,
- iii)  $\rho_X(A) \times Y \subset \rho_Z(A \times Y)$  and  $X \times \rho_Y(B) \subset \rho_Z(X \times B)$  for  $A \in \mathcal{B}_X$  and  $B \in \mathcal{B}_Y$ ,
- iv)  $\mathcal{T}_X \times \mathcal{T}_Y \subset \mathcal{T}_Z$  where  $\mathcal{T}_X \times \mathcal{T}_Y$  means the product topology on  $Z$  of  $\mathcal{T}_X$  and  $\mathcal{T}_Y$ ,
- v)  $\rho_Z$  is strong on  $(Z, \mathcal{T}_X \times \mathcal{T}_Y)$ ,
- vi)  $\rho_Z(f \times 1) = \rho_X(f) \times 1$  and  $\rho_Z(1 \times g) = 1 \times \rho_Y(g)$  for  $f \in M_R^\infty(X)$  and  $g \in M_R^\infty(Y)$ .

Proof. i)  $\rightarrow$  ii) and i)  $\rightarrow$  iii) are obvious. ii)  $\rightarrow$  i) Since  $A \times B = (A \times Y) \cap (X \times B)$  we have  $\rho_Z(A \times B) \subset \rho_X(A) \times \rho_Y(B)$ . Since  $\rho_Z(C^c) =$



$\rho_z(C)^c$  and  $(A \times B)^c = (A^c \times Y) \cap (X \times B^c)$  we have  $\rho_z(A \times B)^c \subset (\rho_x(A) \times \rho_y(B))^c$ . So  $\rho_z(A \times B) = \rho_x(A) \times \rho_y(B)$ . iii)  $\rightarrow$  i) follows similarly from the case ii)  $\rightarrow$  i). Thus it is sufficient to prove i)  $\rightarrow$  iv)  $\rightarrow$  v)  $\rightarrow$  vi)  $\rightarrow$  ii). i)  $\rightarrow$  iv) The topologies  $\mathcal{T}_x$  and  $\mathcal{T}_y$  have bases  $\{\rho_x(A) : A \in \mathcal{B}_x\}$  and  $\{\rho_y(B) : B \in \mathcal{B}_y\}$  respectively. So we can take as a base of  $\mathcal{T}_x \times \mathcal{T}_y$  the family  $\{\rho_x(A) \times \rho_y(B) : A \in \mathcal{B}_x \text{ and } B \in \mathcal{B}_y\}$ . From i)  $\rho_x(A) \times \rho_y(B) = \rho_z(A \times B)$  and  $\rho_z(A \times B)$  is contained in  $\mathcal{T}_z$ , we have then  $\mathcal{T}_x \times \mathcal{T}_y \subset \mathcal{T}_z$ . iv)  $\rightarrow$  v)  $\mathcal{T}_x \times \mathcal{T}_y \subset \mathcal{T}_z$  implies  $C_R^\infty(Z, \mathcal{T}_x \times \mathcal{T}_y) \subset C_R^\infty(Z, \mathcal{T}_z)$ . Since  $C_R^\infty(Z, \mathcal{T}_z) = \{\rho_z(f) : f \in M_R^\infty(Z)\}$  and  $\rho_z(\rho_z(f)) = \rho_z(f)$ ,  $\rho_z$  is strong on  $(Z, \mathcal{T}_x \times \mathcal{T}_y)$ . v)  $\rightarrow$  vi) Since  $\rho_x(f) \times 1$  is  $\mathcal{T}_x \times \mathcal{T}_y$ -continuous, we have  $\rho_z(f \times 1) = \rho_z(\rho_x(f) \times 1) = \rho_x(f) \times 1$ . Similarly  $\rho_z(1 \times g) = 1 \times \rho_y(g)$ . vi)  $\rightarrow$  ii)  $\rho_z(1_A \times 1) = \rho_x(1_A) \times 1$  implies  $\rho_z(A \times Y) = \rho_x(A) \times Y$ , and  $\rho_z(1 \times 1_B) = 1 \times \rho_y(1_B)$  implies  $\rho_z(X \times B) = X \times \rho_y(B)$ .

Q.E.D.

This theorem is similar to [I.T], Ch.5, Th.3 but together with its proof this is more simplified. We note that  $C_R^\infty(Z, \mathcal{T}_x \times \mathcal{T}_y)$  is isomorphic to  $L_R^\infty(X) \otimes_\lambda L_R^\infty(Y)$ , where  $\lambda$  is the least cross norm, and also isomorphic to a closed subalgebra of  $L_R^\infty(Z)$  by  $f \rightarrow \hat{f}$ .

EXAMPLES. 1) Either  $X$  or  $Y$  is purely atomic, i.e. every measurable set of positive measure contains an atom, then there exists a lifting of product form  $\rho_x \times \rho_y$  for every pair of liftings  $\rho_x$  on  $X$  and  $\rho_y$  on  $Y$ .

2) Let  $X=Y$  be the ordinary Lebesgue measure space of  $R$ . Then  $Z$  becomes the ordinary Lebesgue measure space of  $R^2$ . For  $x \in R$ , let  $\mathcal{F}_1(x)$  be the filter basis consisting of the sections of the family  $\{I : \text{bounded open interval in } R \text{ and } x \in I\}$ . For  $(x, y) \in R^2$ , let  $\mathcal{F}_2(x, y)$  be the filter basis consisting of the sections of the family  $\{I \times J : I \text{ and } J \text{ are bounded open intervals in } R \text{ such that } x \in I \text{ and } y \in J\}$ . Then  $\mathcal{F}_1 = \{\mathcal{F}_1(x)\}_{x \in R}$  (respectively  $\mathcal{F}_2 = \{\mathcal{F}_2(z)\}$ )

$z \in R^2$ ) is a weak derivation basis in  $R$  (respectively  $R^2$ ) ([3]).

Put  $\mathcal{J}_{\mathcal{T}_1}(x) = \{\dot{A} \in B(R) : D(A, \mathcal{T}_1)(x) = 0\}$  and  $\mathcal{J}_{\mathcal{T}_2}(z) = \{\dot{A} \in B(R^2) : D(A, \mathcal{T}_2)(z) = 0\}$ . It is easy to show that  $\mathcal{J}_{\mathcal{T}_2}(x) = \mathcal{J}_{\mathcal{T}_1}(x)$  and  $\mathcal{J}_{\mathcal{T}_2}(z) = \mathcal{J}_{\mathcal{T}_2}(z)$ . So  $\mathcal{J}_{\mathcal{T}_2}(x)$  and  $\mathcal{J}_{\mathcal{T}_2}(z)$  are proper ideals of  $B(R)$  and  $B(R^2)$  respectively. Let  $\mathcal{J}_1(\cdot)$  be a mapping from  $R$  into  $\tilde{R}$  such that  $\mathcal{J}_1(x) \supset \mathcal{J}_{\mathcal{T}_1}(x)$  for every  $x \in R$  and  $\rho_1$  be the lifting on  $R$  defined by  $\rho_1(A) = \{x \in R : \dot{A}^c \in \mathcal{J}_1(x)\}$ . We shall show the existence of a lifting of product form  $\rho_1 \times \rho_1$ .

LEMMA 3. For  $(x, y) \in R^2$  denote by  $\mathcal{J}_2((x, y))$  the ideal of  $B(R^2)$  generated by  $\mathcal{J}_1(x) \times \dot{R}$ ,  $\dot{R} \times \mathcal{J}_1(y)$  and  $\mathcal{J}_{\mathcal{T}_2}((x, y))$ . Then  $\mathcal{J}_2((x, y))$  is proper.

Proof. If  $\mathcal{J}_2((x, y))$  isn't proper then there exist  $\dot{A} \in \mathcal{J}_1(x)$ ,  $\dot{B} \in \mathcal{J}_1(y)$  and  $\dot{C} \in \mathcal{J}_{\mathcal{T}_2}((x, y))$  such that  $\dot{R}^2 = (\dot{A} \times \dot{R}) \cup (\dot{R} \times \dot{B}) \cup \dot{C}$ . Consequently  $\dot{A}^c \times \dot{B}^c \in \dot{C}$  and this implies  $\dot{A}^c \times \dot{B}^c \in \mathcal{J}_{\mathcal{T}_2}((x, y))$ . From the definition we have  $D(\dot{A}^c \times \dot{B}^c, \mathcal{T}_2)((x, y)) = \lim_{\mathcal{T}_2((x, y))} \mu_2((\dot{A}^c \times \dot{B}^c) \cap (I \times J)) / \mu_2(I \times J) = \lim_{\mathcal{T}_2((x, y))} \mu_1(\dot{A}^c \cap I) \cdot \mu_1(\dot{B}^c \cap J) / \mu_1(I) \cdot \mu_1(J) = 0$ . So  $D(\dot{A}^c, \mathcal{T}_1)(x) = 0$  or  $D(\dot{B}^c, \mathcal{T}_1)(y) = 0$ . If  $D(\dot{A}^c, \mathcal{T}_1)(x) = 0$  then  $\dot{A}^c \in \mathcal{J}_1(x)$ . This contradicts to  $\dot{A} \in \mathcal{J}_1(x)$  since  $\mathcal{J}_1(x)$  is proper. Similarly  $D(\dot{B}^c, \mathcal{T}_1)(y) = 0$  implies a contradiction.

q.e.d.

Let  $\mathcal{J}(\cdot)$  be a mapping from  $R^2$  into  $\tilde{R}^2$  such that  $\mathcal{J}(z) \supset \mathcal{J}_2(z)$  for every  $z \in R^2$ . Put  $\rho(A) = \{z \in R^2 : \dot{A}^c \in \mathcal{J}(z)\}$  then  $\rho$  is a lifting on  $R^2$ . For  $A \in \mathcal{B}_x$  and  $x \in \rho_1(A)$ ,  $\dot{A}^c \times R \in \mathcal{J}_1(x) \times \dot{R} \subset \mathcal{J}_2((x, y))$  for every  $y \in R^2$ , so  $(x, y) \in \rho(A \times R)$  for every  $y \in R$ . This implies that  $\rho_1(A) \times R \subset \rho(A \times R)$  for every  $A \in \mathcal{B}_x$ . Similarly  $R \times \rho_1(B) \subset \rho(R \times B)$  for every  $B \in \mathcal{B}_y$ . By Theorem 1 we conclude  $\rho$  is of product form  $\rho_1 \times \rho_1$ .

§3. The conditions of the existence of liftings of product form

In §2 we have shown that a lifting of product form  $\rho_x \times \rho_y$  is strong on  $(Z, \mathcal{T}_x \times \mathcal{T}_y)$ , so the existence of a lifting of product

form  $\rho_x \times \rho_y$  is equivalent to the existence of a strong lifting on  $(Z, \mathcal{T}_x \times \mathcal{T}_y)$ . Therefore some results given in [I.T], [4] and [5] are useful to show the conditions of the existence of a lifting of product form. For this we give a new definition. Let  $(X, \mathcal{B}, \mu)$  be a measure space and  $X$  is a topological space with a topology  $\mathcal{T}$ . A (linear) lifting  $\rho$  of  $M_R^\infty(X)$  is called almost strong on  $(X, \mathcal{T})$  if  $C_R^\infty(X, \mathcal{T}) \subset M_R^\infty(X)$  and there exists  $N_0 \in \mathcal{N}_\mathcal{B}$  such that

$$(VII') \quad \rho(f) = f \quad \text{on } X - N_0 \quad \text{for every } f \in C_R^\infty(X, \mathcal{T}).$$

Let  $X, Y, Z, \rho_x$  and  $\rho_y$  be same in §2. Then we prove;

THEOREM 2. The following statements are equivalent to each others.

- i) There exists a lifting on  $Z$  of product form  $\rho_x \times \rho_y$ ,
- ii) There exists an almost strong lifting on  $(Z, \mathcal{T}_x \times \mathcal{T}_y)$ ,
- iii) There exists an almost strong linear lifting on  $(Z, \mathcal{T}_x \times \mathcal{T}_y)$ ,
- iv) There exists a lower density  $\theta$  of  $\mathcal{B}_Z$  such that for some  $N_0 \in \mathcal{N}_{\mathcal{B}_Z}$   $(\rho_x(A) \times Y) - N_0 \subset \theta(A \times Y)$  and  $(X \times \rho_y(B)) - N_0 \subset \theta(X \times B)$  for every  $A \in \mathcal{B}_x$  and  $B \in \mathcal{B}_y$ ,
- v) There exists a weak derivation basis  $\mathcal{F}$  on  $Z$  such that for some  $N_0 \in \mathcal{N}_{\mathcal{B}_Z}$ ,  $D(A \times Y, \mathcal{F}) = 1_{\rho_x(A) \times Y}$  and  $D(X \times B, \mathcal{F}) = 1_{X \times \rho_y(B)}$  on  $Z - N_0$  for every  $A \in \mathcal{B}_x$  and  $B \in \mathcal{B}_y$ .

Proof. i)  $\rightarrow$  ii)  $\rightarrow$  iii) is obvious. iii)  $\rightarrow$  iv) Let  $\rho_z$  be an almost strong linear lifting on  $(Z, \mathcal{T}_x \times \mathcal{T}_y)$  and  $N_0 \in \mathcal{N}_{\mathcal{B}_Z}$  such that  $\rho_z(f) = f$  on  $Z - N_0$  for every  $f \in C_R^\infty(Z, \mathcal{T}_x \times \mathcal{T}_y)$ . For  $A \in \mathcal{B}_x$  put  $\theta(A) = \{z \in Z : \rho_z(1_A)(z) = 1\}$  then  $\theta$  is a lower density of  $\mathcal{B}_Z$  ([I.T]). Since  $1_{\rho_x(A)} \times 1$  is  $\mathcal{T}_x \times \mathcal{T}_y$ -continuous we have  $1_{\theta(A \times Y)} = \rho_z(1_A \times 1) = \rho_z(\rho_x(1_A) \times 1) = \rho_x(1_A) \times 1$  on  $Z - N_0$  for every  $A \in \mathcal{B}_x$ . Consequently  $(\rho_x(A) \times Y) - N_0 \subset \theta(A \times Y)$  for every  $A \in \mathcal{B}_x$ . Similarly  $(X \times \rho_y(B)) - N_0 \subset \theta(X \times B)$  for every  $B \in \mathcal{B}_y$ . iv)  $\rightarrow$  i) Let  $\theta$  be a lower density of  $\mathcal{B}_Z$  which satisfies the conditions in iv). For

$z \in N_0$  denote by  $\mathcal{J}'(z)$  the ideal of  $\mathcal{B}(Z)$  generated by  $\{\dot{A} \times \dot{Y}, \dot{X} \times \dot{B} : z \in \rho_X(A^c) \times Y \text{ and } z \in X \times \rho_Y(B^c)\}$ . Let  $\mathcal{J}(\cdot)$  be a mapping from  $Z$  into  $\tilde{Z}$  such that  $\mathcal{J}(z) \supset \mathcal{J}_0(z)$  for every  $z \in Z - N_0$  and  $\mathcal{J}(z) \supset \mathcal{J}'(z)$  for every  $z \in N_0$ . Then by Proposition 1  $\rho$  is a lifting on  $Z$  and for every  $C \in \mathcal{B}_Z$   $\theta(C) - N_0 \subset \rho(C)$ . For  $A \in \mathcal{B}_X$  and  $z \in (\rho_X(A) \times Y) \cap N_0$  we have  $\dot{A}^c \times \dot{Y} \in \mathcal{J}'(z) \subset \mathcal{J}(z)$  and then  $z \in \rho(A \times Y)$ . Consequently  $\rho_X(A) \times Y \subset \rho(A \times Y)$  for every  $A \in \mathcal{B}_X$ . Similarly  $X \times \rho_Y(B) \subset \rho(X \times B)$  for every  $B \in \mathcal{B}_Y$ . By Theorem 1  $\rho$  is a lifting of product form  $\rho_X \times \rho_Y$ .

i)  $\rightarrow$  v) Let  $\rho_Z$  be a lifting on  $Z$  of product form  $\rho_X \times \rho_Y$ . Put  $N_0 = (\cup \{\rho_Z(A) : A \in \mathcal{B}_Z^*\})^c$  then  $N_0 \in \mathcal{N}_{\mathcal{B}_Z}$ . For  $z \in N_0^c$  denote by  $\mathcal{F}_\rho(z)$  the filter basis consisting of the sections of the family  $\{\rho_Z(A) : z \in \rho_Z(A) \text{ and } A \in \mathcal{B}_Z^*\}$  and for  $z \in N_0$  denote by  $\mathcal{F}_\rho(z)$  an arbitrary filter basis on  $\{\rho_Z(A) : A \in \mathcal{B}_Z^*\}$ . Now for  $A \in \mathcal{B}_Z^*$  and  $z \in \rho_Z(A) - N_0$  there exists  $C_0 \in \mathcal{B}_Z^*$  such  $z \in \rho_Z(C_0)$ . Since the section  $\mathcal{A} = \{\rho_Z(A \cap C) : z \in \rho_Z(A \cap C) \text{ and } C \subset C_0\}$  belongs to  $\mathcal{F}_\rho(z)$  and  $\mu_Z(A \cap C)/\mu_Z(C) = 1$  for every  $C \in \mathcal{A}$ ,  $D(A, \mathcal{F}_\rho)(z) = 1$  on  $\rho_Z(A) - N_0$ . Similarly  $D(A^c, \mathcal{F}_\rho)(z) = 1$  on  $\rho_Z(A^c) - N_0$  and hence  $D(A, \mathcal{F}_\rho)(z) = 1 - D(A^c, \mathcal{F}_\rho)(z) = 0$  on  $\rho_Z(A^c) - N_0$ . We have then  $D(A, \mathcal{F}_\rho)(z) = 1_{\rho_Z(A)}$  on  $Z - N_0$ . This implies  $N_{\mathcal{F}_\rho}^A \in \mathcal{N}_{\mathcal{B}_Z}$  and therefor  $\mathcal{F}_\rho$  is weak. Moreover it satisfies the conditions in v). Finally we shall prove v)  $\rightarrow$  iv). Let  $\mathcal{F}$  be a derivation basis on  $Z$  which satisfies the conditions in v). Let  $\theta_{\mathcal{F}}$  be the lower density of  $\mathcal{B}_Z$  defined in §1. For  $A \in \mathcal{B}_X$  and  $z \in (\rho_X(A) \times Y) - N_0$   $D(A \times Y, \mathcal{F})(z) = 1$ , so  $z \in \theta_{\mathcal{F}}(A \times Y)$ . We have then  $(\rho_X(A) \times Y) - N_0 \subset \theta_{\mathcal{F}}(A \times Y)$ . Similarly  $(X \times \rho_Y(B)) - N_0 \subset \theta_{\mathcal{F}}(X \times B)$ .

Q.E.D.

We shall give some definitions and results for another conditions of the existence of a lifting of product form. Let  $(X, \mathcal{B}, \mu)$  be a measure space. We denote by  $\mathcal{B}^0$  the  $\sigma$ -ring of all  $\sigma$ -finite measurable subsets of  $X$ . For  $A \in \mathcal{B}^*$  denote by  $(A, \mathcal{B} \cap A, \mu|_A)$  the measure space with  $\sigma$ -algebra  $\mathcal{B} \cap A = \{B \cap A : B \in \mathcal{B}\}$  and the mea-

sure  $\mu|_A(B) = \mu(B)$  for  $B \in \mathcal{B} \cap A$ . It is clear that  $\mathcal{B} \cap A$  is identical to the  $\sigma$ -algebra of all  $\mu|_A$ -measurable subsets of  $A$  and  $\mu|_A$  is semi-finite. Since  $\mathcal{B}(A)$  is isomorphic to  $\mathcal{B}(X) \cap A$ ,  $\mathcal{B}(A)$  is complete and  $A$  is localizable. Now let  $X$ ,  $Y$  and  $Z$  be same in §2. We have then  $\mathcal{B}_Z = \{A \subset Z : A \cap (E \times F) \in \mathcal{B}_Z \text{ for every } E \in \mathcal{B}_X^\sigma \text{ and } F \in \mathcal{B}_Y^\sigma\}$  and  $N \in \mathcal{N}_{\mathcal{B}_Z}$  iff  $N \cap (E \times F) \in \mathcal{N}_{\mathcal{B}_Z}$  for every  $E \in \mathcal{B}_X^\sigma$  and  $F \in \mathcal{B}_Y^\sigma$ . Moreover  $(E \times F, \mathcal{B}_Z \cap E \times F, \mu_Z|_{E \times F}) = (E \times F, \overline{\mathcal{B}_X \cap E \times \mathcal{B}_Y \cap F}, \overline{\mu_X|_E \times \mu_Y|_F})$  for every  $E \in \mathcal{B}_X^\sigma$  and  $F \in \mathcal{B}_Y^\sigma$ , where  $(E \times F, \overline{\mathcal{B}_X \cap E \times \mathcal{B}_Y \cap F}, \overline{\mu_X|_E \times \mu_Y|_F})$  is the completion of the usual product measure space of  $(E, \mathcal{B}_X \cap E, \mu_X|_E)$  and  $(F, \mathcal{B}_Y \cap F, \mu_Y|_F)$ . Let  $(X, \mathcal{B}, \mu)$  and  $(X', \mathcal{B}', \mu')$  be measure spaces. A mapping  $\Phi$  from  $X$  onto  $X'$  is called non-singular when  $\Phi$  is invertible bi-measurable and for  $N \in \mathcal{B}$   $N \in \mathcal{N}_{\mathcal{B}}$  iff  $\Phi(N) \in \mathcal{N}_{\mathcal{B}'}$ . Since non-singular mapping preserves  $\sigma$ -finiteness we have the following by above statements.

LEMMA 4. Let  $X$ ,  $X'$ ,  $Y$  and  $Y'$  be measure spaces and  $\Phi_X$  (respectively  $\Phi_Y$ ) be a non-singular mapping from  $X$  onto  $X'$  (respectively  $Y$  onto  $Y'$ ). Let  $Z$  (respectively  $Z'$ ) be the product measure space of  $X$  and  $Y$  (respectively  $X'$  and  $Y'$ ), which is constructed in §2. Put  $\Phi_Z((x, y)) = (\Phi_X(x), \Phi_Y(y))$  for  $(x, y) \in Z$  then  $\Phi_Z$  is a non-singular mapping from  $Z$  onto  $Z'$ .

DEFINITION. Let  $X$  and  $X'$  be measure spaces with liftings  $\rho_X$  and  $\rho'_X$  respectively. We call  $\rho_X$  and  $\rho'_X$  are weakly equivalent and write  $\rho_X \simeq \rho'_X$  when there exist  $N_X \in \mathcal{N}_{\mathcal{B}_X}$ ,  $N'_X \in \mathcal{N}_{\mathcal{B}'_X}$  and a non-singular mapping  $\Phi$  from  $N_X^c$  onto  $N'_X{}^c$  such that

$$(W.E) \quad \Phi(\rho_X(A) - N_X) = \rho'_X(\Phi(A - N_X)) - N'_X \text{ for every } A \in \mathcal{B}.$$

THEOREM 3. Let  $X$ ,  $Y$ ,  $Z$ ,  $\rho_X$  and  $\rho_Y$  be same in §2. Let  $X'$  and  $Y'$  be measure spaces with liftings  $\rho'_X$  and  $\rho'_Y$  such that  $\rho_X \simeq \rho'_X$  and  $\rho_Y \simeq \rho'_Y$  respectively. Let  $Z'$  be the product measure space of  $X'$  and  $Y'$  constructed like as  $Z$ . If there exists a lifting

$\rho_z$  on  $Z$  of product form  $\rho_x \times \rho_y$  then there exists a lifting  $\rho'_z$  on  $Z'$  of product form  $\rho'_x \times \rho'_y$  and  $\rho_z \simeq \rho'_z$ .

Proof. Let  $N_x \in \mathcal{N}_{\mathcal{B}_X}$ ,  $N'_x \in \mathcal{N}_{\mathcal{B}'_X}$ ,  $N_y \in \mathcal{N}_{\mathcal{B}_Y}$ ,  $N'_y \in \mathcal{N}_{\mathcal{B}'_Y}$ ,  $\Phi_x$  and  $\Phi_y$  be sets and mappings by which  $\rho_x \simeq \rho'_x$  and  $\rho_y \simeq \rho'_y$  are defined. In the following denote by  $N_z$  (respectively  $N'_z$ ) instead of  $(N_x^C \times N_y^C)^C$  (respectively  $(N'_x{}^C \times N'_y{}^C)^C$ ). We have then  $N_z \in \mathcal{N}_{\mathcal{B}_Z}$  and  $N'_z \in \mathcal{N}_{\mathcal{B}'_Z}$ . Put  $\Phi((x,y)) = (\Phi_x(x), \Phi_y(y))$  for  $(x,y) \in N_z^C$  then by Lemma 4  $\Phi$  is a non-singular mapping from  $N_z^C$  onto  $N'_z{}^C$ . For  $z' \in N'_z$  denote by  $\mathcal{Q}(z')$  the maximal ideal in  $\mathcal{B}(Z')$  which contains the family  $\{(\dot{A}' \times \dot{Y}'), (\dot{X}' \times \dot{B}') : z' \in \rho'_x(A'^C) \times Y \text{ and } z' \in X' \times \rho'_y(B'^C)\}$ . For  $C' \in \mathcal{B}'_Z$  put  $\rho'_z(C') = \Phi(\rho_z(\Phi^{-1}(C' - N'_z)) - N'_z) \cup \{z' \in N'_z : \dot{C}'^C \in \mathcal{Q}(z')\}$ . It is easy to show that  $\rho'_z$  is a lifting on  $Z'$  of product form  $\rho'_x \times \rho'_y$  and  $\rho_z \simeq \rho'_z$ .

Q.E.D.

To show the corollary of this theorem and the following theorem we introduce the following notations. Let  $(X, \mathcal{B}, \mu)$  be a measure space with a lifting  $\rho$ . For  $A \in \mathcal{B}^*$  we denote by  $\{\rho|_A\}$  the family of all liftings on  $(A, \mathcal{B} \cap A, \mu|_A)$  such that if  $\rho' \in \{\rho|_A\}$  then  $\rho(B) \cap A = \rho'(B) \cap A$  for every  $B \in \mathcal{B} \cap A$ . It is clear that for  $A, B \in \mathcal{B}^*$  such that  $A \equiv B$ , we have  $\rho_1 \simeq \rho_2$  for every  $\rho_1 \in \{\rho|_A\}$  and  $\rho_2 \in \{\rho|_B\}$ . We note that  $\{\rho|_A\} \neq \emptyset$  because for  $B \in \mathcal{B} \cap A$  put  $\rho'(B) = (\rho(B) \cap A) \cup \{x \in A \cap \rho(A)^C : \dot{B}^C \in \mathcal{Q}\}$ , where  $\mathcal{Q}$  is an arbitrary but fixed maximal ideal of  $\mathcal{B}(A)$ , then  $\rho' \in \{\rho|_A\}$ . Moreover if  $A \subset \rho(A)$  then  $\{\rho|_A\}$  is singleton and in this case we write it  $\rho|_A$ . When there is no ambiguity we shall write  $A$  instead of  $(A, \mathcal{B} \cap A, \mu|_A)$ .

COROLLARY. Let  $X, Y, Z, \rho_x$  and  $\rho_y$  be same in §2. For  $A \in \mathcal{B}_X^*$  and  $B \in \mathcal{B}_Y^*$  let  $\rho_A$  and  $\rho_B$  be liftings on  $A$  and  $B$  respectively. If  $\rho_A \in \{\rho_x|_A\}$  and  $\rho_B \in \{\rho_y|_B\}$  and there exists a lifting  $\rho_z$  on  $Z$  of product form  $\rho_x \times \rho_y$  then there exists a lifting  $\rho_{A \times B}$  on  $A \times B$  of product form  $\rho_A \times \rho_B$  and  $\rho_{A \times B} \in \{\rho_z|_{A \times B}\}$ .

Proof. Since  $\rho_A \triangle \rho_x|_{\rho_x(A)}$  and  $\rho_B \triangle \rho_y|_{\rho_y(B)}$ , it is enough to show that there exists a lifting on  $\rho_x(A) \times \rho_y(B)$  of product form  $\rho_x|_{\rho_x(A)} \times \rho_y|_{\rho_y(B)}$ . It is clear that  $\rho_z|_{\rho_x(A) \times \rho_y(B)} = \rho_z|_{\rho_z(A \times B)}$  is the desirous lifting on  $\rho_x(A) \times \rho_y(B)$ .

q.e.d.

Let  $(X, \mathcal{B}, \mu)$  be a measure space with a lifting  $\rho$ . A subfamily  $\{X_\lambda\}_{\lambda \in I} \subset \mathcal{B}$  is called  $\mu$ -dence if  $X_\lambda \cap X_{\lambda'} \in \mathcal{N}_\mathcal{B}$  for  $\lambda \neq \lambda'$  and for every  $A \in \mathcal{B}^*$  there exists  $\lambda \in I$  such that  $A \cap X_\lambda \in \mathcal{B}^*$ . If  $\{X_\lambda\}_{\lambda \in I}$  is  $\mu$ -dence then  $\{\rho(X_\lambda)\}_{\lambda \in I}$  is also  $\mu$ -dence. Moreover put  $N = (\bigcup_{\lambda \in I} X_\lambda)^c$  then  $N \in \mathcal{N}_\mathcal{B}$  and for  $A, B \in \mathcal{B}$   $A \equiv B$  iff  $A \cap X_\lambda \equiv B \cap X_\lambda$  for every  $\lambda \in I$ .

THEOREM 4. Let  $X, Y, Z, \rho_x$  and  $\rho_y$  be same in §2. If there exists a  $\mu_x$ -dence family  $\{X_\lambda\}_{\lambda \in I}$  (respectively  $\mu_y$ -dence family  $\{Y_\kappa\}_{\kappa \in K}$ ) such that for every  $\lambda \in I$  there exists a lifting  $\rho_\lambda$  on  $X_\lambda$  which satisfies  $\rho_\lambda \in \{\rho_x|_{X_\lambda}\}$  (respectively for every  $\kappa \in K$  there exists a lifting  $\rho_\kappa$  on  $Y_\kappa$  which satisfies  $\rho_\kappa \in \{\rho_y|_{Y_\kappa}\}$ ) and if there exists a lifting  $\rho_{(\lambda, \kappa)}$  on  $X_\lambda \times Y_\kappa$  of product form  $\rho_\lambda \times \rho_\kappa$ . Then there exists a lifting  $\rho_z$  on  $Z$  of product form  $\rho_x \times \rho_y$  and  $\rho_{(\lambda, \kappa)} \in \{\rho_z|_{X_\lambda \times Y_\kappa}\}$ .

Proof. By the assumption, Theorem 3 and statements following to it there exists a lifting  $\rho'_{(\lambda, \kappa)}$  on  $\rho_x(X_\lambda) \times \rho_y(Y_\kappa)$  of product form  $\rho_x|_{\rho_x(X_\lambda)} \times \rho_y|_{\rho_y(Y_\kappa)}$ . Put  $N = ((\bigcup_{\lambda \in I} \rho_x(X_\lambda)) \times (\bigcup_{\kappa \in K} \rho_y(Y_\kappa)))^c$  then  $N \in \mathcal{N}_{\mathcal{B}_Z}$ . Now for every  $z \in N$ , we denote by  $\mathcal{J}(z)$  the maximal ideal of  $\mathcal{B}(Z)$  which contains the family  $\{\dot{A} \times \dot{Y}, \dot{X} \times \dot{B} : z \in \rho_x(A^c) \times Y \text{ and } z \in X \times \rho_y(B^c)\}$ . For  $A \in \mathcal{B}_Z$ , put  $\rho_z(A) = \bigcup_{(\lambda, \kappa) \in I \times K} \rho'_{(\lambda, \kappa)}(A \cap (\rho_x(X_\lambda) \times \rho_y(Y_\kappa))) \cup \{z \in N : \dot{A}^c \in \mathcal{J}(z)\}$ , then  $\rho_z$  is a lifting on  $Z$  of product form  $\rho_x \times \rho_y$  and  $\rho_{(\lambda, \kappa)} \in \{\rho_z|_{X_\lambda \times Y_\kappa}\}$ .

Q.E.D.

## References

- [I.T] Ionescu Tulcea, A. and C. TOPICS IN THE THEORY OF LIFT-  
ING. Berlin-Heidelberg-New York. Springer-Verlag 1969
1. Berberian, S.K. MEASURE AND INTEGRATION. New York. Macmillan.
  2. Fillmore, P.A. On Topology Induced by Measure. Proc. Amer. Math. Soc. Vol. 17 (1966) p.p. 854-857.
  3. Hayes, C.A. and C.Y. Pauc. DERIVATION AND MARTINGALES. Berlin-Heidelberg-New York. Springer-Verlag 1970
  4. Ionescu Tulcea, A. and C. On Liftings and Derivation Bases. J. Math. Anal. and Appl. Vol. 35 (1971) p.p. 449-466.
  5. Köizōw, D. DIFFERENTIATION VON MASSEN. Lecture Notes in Mathematics. Band 65. Springer-Verlag 1968
  6. Ryan, R. Representative Sets and Direct Sums. Proc. Amer. Math. Soc. Vol 15 (1964) p.p. 387-390.